Performance evaluation of modulation methods: a combinatorial approach

D. Krob *, E.A. Vassilieva †

1 Introduction

Modulating a numeric signal corresponds to the fact of transforming the digital sequence that represents it, into a wave form. Modulation is therefore clearly a technique of main interest in a number of ingeneering domains such as computer networks, mobile communications, satellite transmissions, television diffusion, etc.

Due to their practical importance, modulation methods were therefore widely studied in signal processing. The classical Proakis textbook devotes for instance a full chapter to this subject (cf Chapter 5 of [9]). One of the most important problem in this area is to be able to design and to evaluate the performance characteristics of the optimum receivers associated with a given modulation method. The performance analyses that occur in such a context, reduce in particular to the computation of various probability errors (see again [9] for more details).

Among the different modulation protocols, a rather important (in practice) class consists in methods where the modulation references (i.e. the wave forms associated with all possible digital sequences of a given length) are also modulated and hence submitted to the transmission noise. In this kind of situation, the demodulating decision needs by consequence to account two noisy informations (the transmitted signal and the transmitted references). The computation of the probability errors appearing in such contexts, involves therefore very often to compute the following type of probability:

$$P(U < V) = P\left(\sum_{j=1}^{N} u_j^* u_j \; < \; \sum_{j=1}^{N} v_j^* v_j \right) , \tag{1}$$

where the u_i and v_i's stand for independent complex Gaussian random variables with arbitrary variances respectively denoted

$$E[u_j^* u_j] = \chi_j \, , \quad E[v_j^* v_j] = \delta_j \tag{2}$$

for every $j \in [1, N]$ (see Section 3.1 for more details).

The problem of computing explicitely this last probability was hence studied by a number of researchers coming from signal processing (cf [2, 6, 9, 10]). The most interesting result in

* LIAFA (CNRS) - Université Paris 7 - 2, place Jussieu - 75251 Paris Cedex 05 – France – e-mail: dk@liafa.jussieu.fr

† LIAFA (CNRS) - Université Paris 7 - 2, place Jussieu - 75251 Paris Cedex 05 – France – e-mail: katya@liafa.jussieu.fr

this direction was obtained by Barett (cf [2]) who proved that one can express the probability defined by (1) as follows:

$$P(U < V) = \sum_{k=1}^{N} \left(\prod_{j \neq k} \frac{1}{1 - \delta_k^{-1} \delta_j} \prod_{j=1}^{N} \frac{1}{1 + \delta_k^{-1} \chi_j} \right) . \tag{3}$$

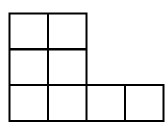
It appears that this last formula can in fact be interpreted in a purely combinatorial way, using Schur functions and Young tableaux (see Section 3.1 for the details). This new approach already lead to the obtention of the first, both algorithmically efficient and numerically stable, practical method for computing the probability $P(U < V)$ (see again Section 3.1 or [3, 4] for more informations).

In this paper, we continue the combinatorial study of Barett's formula by showing that it is in fact highly connected with a slight modification of a very classical bijection of Knuth (cf [7] or [5] for a more recent presentation) between pairs of Young tableaux of conjugated shapes and $\{0, 1\}$-matrices. These new considerations give us clearly a better understanding of Barett's result. They also allowed us to obtain the first results with respect to specializations of Barett's formula that were still not known for the moment (see Section 4.4).

2 Background

2.1 Partitions

A *partition* is a finite nondecreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of positive integers. Graphically each such partition can be represented by a diagram of $\lambda_1 + \ldots + \lambda_m$ boxes, called its *Ferrers diagram*, whose $(m - i + 1)$-th row contains λ_i boxes for every $1 \leq i \leq n$. The partition $\lambda = (2, 2, 4)$ is for instance represented by the Ferrers diagram



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A given partition is then called the *shape* of the associated Ferrers diagram.

Using this graphic representation, one can easily define the notion of *conjugated partition*. The conjugated partition $\lambda\tilde{}$ of a given partition λ is indeed just the partition obtained by reading the heights of the columns of the Ferrers diagram associated with λ. One has here for instance $\lambda\tilde{} = (1, 1, 3, 3)$ when $\lambda = (2, 2, 4)$ as it can be seen on the previous picture.

When λ is a partition whose Ferrers diagram is contained into the square (N^N) with N rows (of length N), one can also associate with it its *complementary* partition, denoted by $\overline{\lambda}$, which is the conjugate of the partition ν whose Ferrers diagram is the complement (read from top to bottom) of the Ferrers diagram of λ in the square (N^N). For instance, for $N = 6$ and $\lambda = (1, 1, 2, 3)$, we have $\nu = (3, 4, 5, 5, 6, 6)$ and $\overline{\lambda} = (2, 4, 5, 6, 6, 6)$ as it can be checked on the following Figure 1. The Ferrers diagram associated with λ is here represented by the boxes filled with $\bullet$ and the boxes filled with $\diamond$ correspond in the same way to the partition ν (that can be obtained by computing the number of such boxes per row) or to the complementary partition $\overline{\lambda}$ (that can be obtained by computing the number of such boxes per column).

We will call *tabloid* of shape λ the filling of a Ferrers diagram of shape λ with arbitrary positive integers. A filling of the boxes of a Ferrers diagram of shape λ with positive integers is

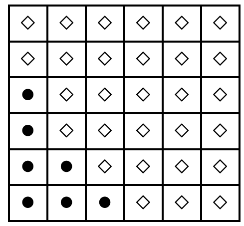

Figure 1: Two complementary Young tableaux.

called a *Young tableau* (of shape λ) whenever the numbers are weakly increasing along all rows and strictly increasing along all columns. For example, the diagram

3	5		
2	2		
1	1	1	4

is a Young tableau of shape $(2,2,4)$.

Let $X = \{\, x_i,\ 1 \le i \le n \,\}$ be a set of n variables. One associates then to any Young tableau T filled by integers not greater than n, a monomial X^T defined as the product of the factors x_i for each entry i of T. For the Young tableau T of the above example, one has for instance

$$X^T = x_1^3\, x_2^2\, x_3\, x_4\, x_5$$

if we set $X = \{\, x_1, x_2, x_3, x_4, x_5\}$. The *Schur function* $s_\lambda(X)$ associated with the partition λ is then defined as the sum of the monomials X^T, for T running over all Young tableaux of shape λ, filled with numbers not greater than n. We recall that each Schur function is a symmetric polynomial over X and that the Schur functions are a linear basis of the algebra of symmetric polynomials over X (for more informations on these questions, the reader should refer to the classical textbook [8]).

2.2 Gaussian polynomials and the $q-$Newton formula

Let q be a variable. Then the expression

$$[n]_q = 1 + q + q^2 + \ldots + q^{n-1} = \frac{1-q^n}{1-q}.$$

is called the $q-$integer of order n (this notation comes from the fact that the specialization of a q-integer at $q = 1$ gives of course the usual corresponding integer). We recall that the $q-$factorial is then defined by

$$[n!]_q = [1]_q\,[2]_q \ldots [n]_q = \frac{\displaystyle\prod_{i=1}^{n}(1-q^i)}{(1-q)^n}\ .$$

Finally the expression

$$\left(\begin{array}{c} n \\ m \end{array}\right)_q = \frac{[n!]_q}{[m!]_q\,[(n-m)!]_q}$$

is known as the Gaussian polynomial of order (n,m). It is clearly the q-analogue of the usual binomial coefficient of the same order. We refer to [1] for more informations about Gaussian polynomials.

We will however recall the $q-$Newton formula (see [1]):

$$\sum_{j=0}^{N} \left(\begin{array}{c} N \\ j \end{array} \right)_q (-1)^j \, q^{\frac{j(j-1)}{2}} \, z^j = \prod_{k=0}^{N-1} (1 - z\, q^k) \ . \tag{4}$$

Note that the q-Newton formula specializes for $z = q$ to

$$\sum_{j=0}^{N} \left(\begin{array}{c} N \\ j \end{array} \right)_q (-1)^j \, q^{\frac{j(j+1)}{2}} = \prod_{k=1}^{N} (1 - q^k) \ . \tag{5}$$

2.3 Column bumping process and Knuth bijection

The column bumping (or column-insertion) and row bumping (or row-insertion) processes are classical dual constructions that provide algorithms to transform a given word on the alphabet $\mathbb{N}$ of all integers into a Young tableau. Here we will give some brief informations on the mechanism of column bumping and present a famous bijection of Knuth, between pairs of Young tableaux and $\{0, 1\}$-matrices, (cf [7]) which is, being a variation on the well known Robinson-Schensted correspondence, based on column bumping process in its construction. For a better overview of the subject, we refer to [5].

The column bumping process is organized as follows. Take a positive integer x and a Young tableau T. Put x in a new box at the top of the first column if it is strictly larger than all the entries of the column. If it is not the case, bump the lowest (i.e., the smallest) entry in the column that is greater than or equal to x and replace it by x. Move the bumped entry to the top of the next column if possible, or recursively bump one of the elements to the next column otherwise. The process continues until the bumped entry can go at the top of the next column, or until it becomes the only entry of a new column. Note, that the bumping here takes place in a zig-zag path that moves to the right, never moving up, and the result is always another tableau. If the location of the box that is added is known, the process can be reversed.

We are now in position to present the one-to-one correspondence (due to Knuth) between matrices M whose entries are zeros and ones (or equivalently two-rowed arrays without repeated pairs) and pairs (P, Q) of Young tableaux with conjugated shapes. The construction of this bijection can be reflected in the following steps.

1. Associate first with M the array

   $$A = \left(\begin{array}{ccccc} u_1 & u_2 & \ldots\ldots & u_r \\ v_1 & v_2 & \ldots\ldots & v_r \end{array} \right),$$

   that consists of all the indices (classified in the lexicographic order) corresponding to the 1-entries of M (all the entries u_i of the first row are therefore in weakly increasing order and one has moreover $v_{i-1} \leq v_i$ in the second row whenever one has $u_{i-1} = u_i$ in the first).

2. Perform column bumping with all the variables v_i of the second row of the array A begining from the first variable v_1 and moving one by one to the very last variable v_r. The result is a Young tableau P.

3. The second Young tableau Q is just an encoding of the order in which the first Young tableau P was constructed on the previous step. We first place the first element u_1 in the (conjugated of the) first box that appeared during the column bumping process that was used to construct P. The second element u_2 is placed in the same way in the box which is conjugated to the second box that appeared in this process, etc.

Applying the reversed column bumping process to the tableau P and removing in the same time the corresponding boxes of the tableau Q, allows us to reconstruct the initial array A by writing down in the order of their appearing the bumped out entries.

2.4 Columns and their complements

In this subsection, we will finally pay some attention to *columns*, i.e. to Young tableaux of shape $1^k = (1, \ldots, 1)$. The number k of 1's is here equal to the length of the column. We will be only interested by columns of length less than some positive number N, filled with integers belonging to the set $\{1, \ldots, N\}$.

Introduce now some new notations. Let $I = \{\, i_1 < i_2 < \ldots < i_l \,\}$ be a strictly increasing subsequence of $\{1, \ldots, N\}$. Denote then by $c\,(I)$ the column of length l filled with all integer of I, increasing from bottom to top. We will also use in the sequel the notation $c\,(I, J)$ to denote the column of length $l+m$ filled with the elements of the sets $I = \{i_1, i_2, \ldots, i_l\}$ and $J = \{j_1, j_2, \ldots, j_m\}$, if the sequence

$$i_1 < i_2 < \cdots < i_l < j_1 < j_2 < \cdots < j_m$$

is a strictly increasing subsequence of $\{1, \ldots, N\}$.

Let $K = \{k_1 < k_2 < \ldots < k_t\}$ be a strictly increasing subsequence of positive integers. The column $c\,(K)$ is then called the *complement* (within $\{1, \ldots, N\}$) of the column $c\,(I)$ if we have $K = \{1, \ldots, N\} \setminus I$. In the sequel, this column will be denoted

$$c\,(K) = \overline{c\,(I)}\ .$$

Let us again take two sets $I = \{i_1, i_2, \ldots, i_l\}$ and $J = \{j_1, j_2, \ldots, j_m\}$. We will say that the column $c\,(I)$ is *less or equal* than the column $c\,(J)$ and write $c\,(I) \preceq c\,(I)$ if one has $m \leq n$ and $i_k \leq j_k$ for every $1 \leq k \leq m$. In other words, a column $c\,(I)$ is less or equal to a column $c\,(J)$ if and only if one obtains a Young tableau when putting the column associated with J at the right of the column associated with I.

3 Performance analysis of modulation protocols

3.1 Barret's formula

The analysis of many practical digital transmission systems involves the computation of the probability that a given Hermitian quadratic form in complex normal variates is negative. Numerous such examples can be found in Proakis's standard textbook (cf [9]). This kind of problem appears in particular in the context of performance analysis of classical demodulation protocols acting on modulated signals transfered on noisy Gaussian channels.

A first expression for the probability that a given Hermitian quadratic form in complex normal variates is negative, was first derived by Turin (see [10]) and used later by Barrett (see [2]) to unveil a closed form expression for this probability as a rational function of the eigenvalues of the corresponding covariance matrix. Barett showed indeed that the general problem discussed above can be reduced to the study of the probability $P(U < V)$ presented in the first section of this paper (cf formula (1) of section 1).

Barett gave also the explicit formula (3) that was, up to this year, the best known approach from computing the probability $P(U < V)$ from a practical point of view. Alternate methods involving either direct contour integration of the associated characteristic function along a

carefully selected path so as to maximise numerical stability, or algebraic manipulations like in [9] (Annex B) or [6], provide other approaches involving numerical quadrature of trigonometric functions.

All these methods lead however to algorithms that are not numerically stable due to the presence of artificial singularities (such as the singularities $\delta_i = \delta_j$ of Barett's formula (3)). It is therefore important to notice that the first efficient and stable method for computing $P(U < V)$ was very recently proposed by Dornstetter, Krob and Thibon (cf [3] or section 3.2), based initially on symmetric functions techniques. We recall below their algorithm for the sake of completeness (cf [3, 4] for all details).

- `Step 1.` Consider the two polynomials defined by setting

$$X(z) = \prod_{i=1}^{N} (1 - \chi_i\, z) \ , \qquad \Delta(z) = \prod_{i=1}^{N} (1 + \delta_i\, z) \ .$$

- `Step 2.` Compute the unique polynomial π of degree less or equal $N-1$ such that one has

$$\pi(z)\, X(z) + \mu(z)\, \Delta(z) = 1$$

  where μ stands for some other polynomial of degree less or equal to $N-1$.

- `Step 3.` Evaluate $P(U < V) = \pi(0)$.

The algorithmic efficiency and the numerical stability of this result comes then just from the fact that the second step of the above method can be made using the generalized Euclidean algorithm which is a very classical method which has the two above mentionned properties.

3.2 The combinatorial version of Barett's formula

Barett's formula in fact can be rewritten as a rational fraction, i.e.

$$P(U < V) = \frac{F(\chi, \delta)}{\displaystyle\prod_{1 \le i,j \le N} (\chi_i + \delta_j)} \ , \tag{6}$$

where $F(\chi, \delta)$ is a symmetric polynomial with respect to the χ_i and to the δ_j. Moreover can be proved (cf [4]) that $F(\chi, \delta)$ can be expressed in terms of Schur functions, i.e.

$$F(\chi, \delta) = \sum_{\lambda \subseteq (N^{N-1})} s_{(\lambda, N)}(\{\delta_1, \ldots, \delta_N\})\, s_{\lambda^\vee}(\{\chi_1, \ldots, \chi_N\}) \ , \tag{7}$$

where $\lambda^\vee$ denotes the complement of the partition (λ, N) within the rectangle N^N.

Note now that each monomial that appear in the right hand side of equation (7) can be obtained by taking the product of all the elements of a square tableau of shape $N \times N$ consisting in two Young tableaux of complementary shapes (i.e. as given by Figure 1 of Section 2.1) that respect the two following constraints:

- **Condition S1**: the first tableau is only filled by variables that belong to the alphabet $\delta = \{\, \delta_1, \ldots, \delta_N \,\}$ and *the length of its first row is equal to* N,

- **Condition S2**: the second tableau is only filled by variables that belong to the alphabet $\chi = \{\chi_1, \ldots, \chi_N\}$.

A typical example of such a combinatorial structure is given in Figure 2. Note that the first tableau is written here in the usual way. On the other hand, the second tableau is organized a bit differently: its rows (resp. its columns) are placed from top to bottom (resp. from right to left) in the space corresponding to the complement of the first tableau within the square $N \times N$.

χ_6	χ_5	χ_4	χ_3	χ_2	χ_1
δ_4	χ_6	χ_5	χ_4	χ_2	χ_1
δ_4	δ_5	δ_6	χ_4	χ_3	χ_2
δ_3	δ_3	δ_5	χ_5	χ_4	χ_3
δ_2	δ_2	δ_3	δ_4	χ_4	χ_3
δ_1	δ_1	δ_2	δ_2	δ_2	δ_3

Figure 2: A typical example of complementary filling of a square tableau.

We will now proceed exploring the polynomial $F(\chi, \delta)$, involved in formula (6) and given by formula (7), by calculating the number α_N of all square $N \times N$ tableaux filled as in the typical example of Figure 2. Knowing this last integer will give us the exact algorithmic complexity of the formula (6). One should indeed just notice that α_N is equal to the number of distinct monomials involved in $F(\chi, \delta)$, from which one can easily deduce that the complexity of the computation of $F(\chi, \delta)$ is exactly equal to $N^2 \alpha_N$.

It appears unfortunately that $\alpha_N n = 2^{N^2-1}$, as proved in the next result, which implies that formula (6) can not be used in practice as soon as N grows. The combinatorial formula (6) is however absolutely not useless (from a theoretical point of view) since it can be reformulated equivalently in the terms of the algorithm given at the end of Section 3.1, which is both practically very efficient (its complexity is quadratic as Barett's formula) and numerically stable as already stated (cf [3, 4] for all details).

Proposition 3.1 *The number α_N of square tableaux of shape $N \times N$ filled by two complementary Young tableaux satisfying to conditions* **S1** *and* **S2** *is given by the formula:*

$$\alpha_N = 2^{N^2-1} .$$

Proof – Let us first notice that the conjunction of formulas (3) and (6) shows that one has:

$$\frac{F(\chi, \delta)}{\prod_{1 \le i,j \le N} (\chi_i + \delta_j)} = \sum_{k=1}^{N} \left(\prod_{j \neq k} \frac{1}{1 - \delta_k^{-1} \delta_j} \prod_{j=1}^{N} \frac{1}{1 + \delta_k^{-1} \chi_j} \right) . \tag{8}$$

Let us begin by replacing everywhere χ_i and δ_i by t^i in this last formula. This simple trick will allow us to avoid the singularities of Barett's formula, corresponding to the situation when some of the δ_i's collapse to a common value. Note that this replacement of variables transforms the symmetric function $F(\chi, \delta)$ into a polynomial $P(t)$ that provides the desired number α_N when t equal 1. Therefore it is sufficient to calculate $P(1)$ to get the value of α_N.

Note now that formula (8) gives us immediately the following expression for $P(t)$:

$$P(t) = \prod_{1 \le i,j \le N} \left(t^i + t^j\right) \left(\sum_{k=1}^{N} \left(\prod_{1 \le j \neq k \le N} \frac{1}{1 - t^{j-k}} \prod_{j=1}^{N} \frac{1}{1 + t^{j-k}} \right) \right) . \tag{9}$$

It appears that one can prove that the identity

$$\sum_{k=1}^{N} \left(\prod_{1 \le j \neq k \le N} \frac{1}{1 - t^{j-k}} \prod_{j=1}^{N} \frac{1}{1 + t^{j-k}} \right) = \frac{1}{2} \tag{10}$$

holds for every t (see Lemma 3.2 below). Hence one gets

$$P(t) = \frac{1}{2} \left(\prod_{1 \le i,j \le N} (t^i + t^j) \right) ,$$

from which one can immediately conclude that $\alpha_N = P(1) = 2^{N^2-1}$. Hence the proof of our proposition now reduces to the proof of the following lemma.

Lemma 3.2 *For every t, one has:*

$$\sum_{k=1}^{N} \left(\prod_{1 \le j \neq k \le N} \frac{1}{1 - t^{j-k}} \prod_{j=1}^{N} \frac{1}{1 + t^{j-k}} \right) = \frac{1}{2} . \tag{11}$$

Proof – We will perform a number of equivalent transformations of identity (11) in order to reduce it into a classical identity, which will finish our proof.

Taking first into account that

$$\prod_{j=1}^{N} \frac{1}{1 + t^{j-k}} = \frac{1}{2} \left(\prod_{1 \le j \neq k \le N} \frac{1}{1 + t^{j-k}} \right) ,$$

we can rewrite equation (11) in the equivalent way:

$$\sum_{k=1}^{N} \left(\prod_{1 \le j \neq k \le N} \frac{1}{1 - t^{j-k}} \right) \left(\prod_{1 \le j \neq k \le N} \frac{1}{1 + t^{j-k}} \right) = \sum_{k=1}^{N} \prod_{1 \le j \neq k \le N} \frac{1}{1 - t^{2(j-k)}} = 1 . \tag{12}$$

Let us further develop the left hand side of the last equation. We then get

$$\sum_{k=1}^{N} \prod_{1 \le j \neq k \le N} \frac{1}{1 - t^{2(j-k)}} = \sum_{k=1}^{N} \prod_{j=1}^{k-1} \frac{1}{1 - t^{2(j-k)}} \prod_{j=k+1}^{N} \frac{1}{1 - t^{2(j-k)}}$$

$$= \sum_{k=1}^{N} \prod_{j=1}^{k-1} \frac{1}{1 - t^{(-2j)}} \prod_{j=1}^{N-k} \frac{1}{1 - t^{2j}} = \sum_{k=1}^{N} \prod_{j=1}^{k-1} \frac{(-1)\, t^{2j}}{1 - t^{2j}} \prod_{j=1}^{N-k} \frac{1}{1 - t^{2j}}$$

$$= \sum_{k=1}^{N} \frac{(-1)^{k-1}\, t^{2(1+2+\ldots+(k-1))}}{\prod_{j=1}^{k-1} (1 - t^{2j}) \prod_{j=1}^{N-k} (1 - t^{2j})} = \sum_{k=1}^{N} \frac{(-1)^{k-1}\, t^{k\,(k-1)}}{\prod_{j=1}^{k-1} (1 - t^{2j}) \prod_{j=1}^{N-k} (1 - t^{2j})} .$$

Substituting $t^2 = q$ in the previous identity allows us therefore to rewrite identity (12) into the following alternate form:

$$\sum_{k=1}^{N} \frac{(-1)^{k-1}\, q^{\frac{k(k-1)}{2}}}{\prod_{j=1}^{k-1} (1-q^j) \prod_{j=1}^{N-k} (1-q^j)} = 1. \tag{13}$$

Multiplying now both parts on this last identity by the polynomial $[(N-1)!]_q\,(1-q)^{N-1}$ and using the definition of the Gaussian polynomials, we can rewrite identity (13) as

$$[(N-1)!]_q\,(1-q)^{N-1} = \sum_{k=1}^{N} \frac{(-1)^{k-1}\, q^{\frac{k(k-1)}{2}}\, [(N-1)!]_q}{[(k-1)!]_q\,[(N-k)!]_q} = \sum_{k=1}^{N} \binom{N-1}{k-1}_q (-1)^{k-1}\, q^{\frac{k(k-1)}{2}} .$$

This last formula can therefore be equivalently rewritten as

$$\prod_{i=1}^{N-1} (1-q^i) = \sum_{j=0}^{N-1} \binom{N-1}{j}_q (-1)^j\, q^{\frac{j(j+1)}{2}} ,$$

which is exactly the well known $q-$Newton formula (see Section 2.2 or [1]). Hence the initial identity is true since it is just a transformation of this last classical identity. ■

4 A bijective proof of the combinatorial formula

The previous proof gave us the desired number α_N of monomials involved in $F(\chi, \delta)$ in a purely analytic way. It however did not provide any insight, neither in the structure of $F(\chi, \delta)$, nor in the simplicity of our result since the fact that $\alpha_N = 2^{N^2-1}$ is indeed clearly remarcable.

We will devote now this section to the construction of a bijective proof of this last result. It will appear in fact that this construction will also help us in studying a number of specializations of Barett's formula. Hence our bijective proof will be rather interesting both from a theoretical and a practical point of view.

4.1 A more general structure

In order to prove that $\alpha_N = 2^{N^2-1}$ in a bijective way, we will introduce a slightly generalized version of the combinatorial structures that were involved in the description of $F(\chi, \delta)$. These new combinatorial structures will just consist in the set, that we will denote by $\mathcal{T}_N$, of all $N \times N$ squares divided into two complementary Young tableaux (without any constraint on them) respectively filled by elements of the alphabets δ and χ. The following picture shows two typical examples of an element of $\mathcal{T}_6$.

Note that the first tableau is again written in the usual way. On the other hand, the second tableau is organized again differently: its rows (resp. its columns) are placed from top to bottom (resp. from right to left) in the space corresponding to the complement of the first tableau within the square $N \times N$.

We will prove bijectively in the sequel that the cardinality of $\mathcal{T}_N$ is equal to 2^{N^2}. This will immediately imply that $\alpha_N = 2^{N^2-1}$ due to the fact that the number of elements of $\mathcal{T}_N$ whose first tableau has a first row of length N is clearly equal to the number of elements of $\mathcal{T}_N$ whose

χ_6	χ_5	χ_4	χ_3	χ_2	χ_1
δ_4	χ_6	χ_5	χ_4	χ_2	χ_1
δ_4	δ_5	δ_6	χ_5	χ_2	χ_1
δ_3	δ_3	δ_4	χ_6	χ_2	χ_1
δ_2	δ_2	δ_2	δ_2	χ_2	χ_1
δ_1	δ_1	δ_1	δ_1	δ_1	δ_1

δ_6	χ_5	χ_4	χ_3	χ_2	χ_1
δ_4	χ_6	χ_5	χ_4	χ_2	χ_1
δ_4	δ_5	δ_6	χ_4	χ_3	χ_2
δ_3	δ_3	δ_5	χ_5	χ_4	χ_3
δ_2	δ_2	δ_3	δ_4	χ_4	χ_3
δ_1	δ_1	δ_2	δ_2	δ_2	χ_4

Figure 3: Two typical elements of $\mathcal{T}_6$.

second tableau has a first row of length N (which corresponds to the case where the first tableau has a first row of length strictly less than N).

To get this last result, we will construct a bijection – presented in the next subsection – between $\mathcal{T}_N$ and the set $\mathcal{M}_{N\times N}(\{0,1\})$ of all $\{0,1\}$-matrices of size $N \times N$.

4.2 Construction of the bijection

We will now present our bijection between $\mathcal{M}_{N\times N}(\{0,1\})$ and $\mathcal{T}_N$. Our construction is based on a slight variation of the well known Knuth's bijection, presented in Section 2.3. We will see in the sequel that it has some deep and not obvious symmetry properties that will be fundamental for highlighting Barett's formula in a totally new way.

Let therefore M be a matrix of $\mathcal{M}_{N\times N}(\{0,1\})$. We associate then with M the word $w(M)$ over the alphabet $\{1,2,\ldots,N\} \times \{1,2,\ldots,N\}$ defined as follows.

1. Construct first the 2-row array A_N which is equal to the sequence of the N^2 pairs (i,j) of $\{1,2,\ldots,N\} \times \{1,2,\ldots,N\}$ taken in the lexicographic order, i.e.

$$A_N = \begin{pmatrix} 1 & \ldots & 1 & 2 & \ldots & 2 & \ldots\ldots & N & \ldots & N \\ 1 & \ldots & N & 1 & \ldots & N & \ldots\ldots & 1 & \ldots & N \end{pmatrix} .$$

2. Select then in this array all the entries that correspond to the 1's of of M. We obtain then a word $w(M)$ on the alphabet $\{1,2,\ldots,N\} \times \{1,2,\ldots,N\}$ by reading all these entries from left to right.

Example 4.1 *Let us consider the matrix*

$$M = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} .$$

Then one has

$$A_3 = \begin{pmatrix} 1 & 1 & \boxed{1} & \boxed{2} & 2 & 2 & 3 & \boxed{3} & \boxed{3} \\ 1 & 2 & \boxed{3} & \boxed{1} & 2 & 3 & 1 & \boxed{2} & \boxed{3} \end{pmatrix}$$

where we squared the entries associated with the 1*'s of* M*. Hence we get*

$$w(M) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} .$$

We apply now Knuth's bijection to $w(M)$ in order to get two Young tableaux

$$(T_1, T_2)$$

of conjugated shapes λ_1 and $\lambda_1\tilde{\ }$. We will now associate with the tableau T_2 a new tableau $\overline{T_2}$ of shape $\overline{\lambda_1}$ (the complementary partition of λ_1 within $N \times N$) that is constructed as follows. Note first that one defines a unique tabloid $\overline{T_2}$ of shape $\overline{\lambda_1}$ by asking (for every $i \in [1, N]$) that the i-th column of $\overline{T_2}$ consists exactly of all the letters of $\{ 1, \ldots, N \}$ that do not appear in the $(N-i+1)$-th column of T_2. It appears that this tabloid is in fact a Young tableau.

Proposition 4.2 *The tabloid $\overline{T_2}$ is a Young tableau.*

Proof – The proof will be made in three steps. In all lemmas that are involved in this proof, we will use the notations and definitions of Section 2.4. The proof of the two first lemmas will be found in the final version of this paper.

Lemma 4.3 *Let $c\,(I, J)$ and $c\,(I)$ be the two columns such that $c\,(I, J) \preceq c\,(I)$. Then, for their complements $\overline{c\,(I, J)}$ and $\overline{c\,(I)}$ holds the following unequality:*

$$\overline{c\,(I)} \preceq \overline{c\,(I, J)}\ .$$

Lemma 4.4 *Let $c\,(I)$ and $c\,(J)$ be two columns of the same length such that $c\,(I) \preceq c\,(J)$. Then, for their complements $\overline{c\,(I)}$ and $\overline{c\,(J)}$, holds the following inequality:*

$$\overline{c\,(J)} \preceq \overline{c\,(I)}\ .$$

Proposition 4.2 is now an immediate consequence of the next (and last) lemma.

Lemma 4.5 *Let $c(I, J)$ and $c(K)$ be two columns that satisfy the inequality $c\,(I, J) \preceq c\,(K)$. Suppose also that the two subsets I and K of $\{1, \ldots, N\}$ have the same number of elements. Then, for the complements $\overline{c\,(I, J)}$ and $\overline{c\,(K)}$ of the two above columns, one has:*

$$\overline{c\,(K)} \preceq \overline{c\,(I, J)}\ .$$

Proof – The statement of our lemma can be easily obtained by applying Lemma 4.3 and Lemma 4.4 in order to get the inequalities:

$$c\,(I, J) \preceq c\,(I) \preceq c\,(K)\ .$$

This ends therefore both the proof of our lemma and of Proposition 4.2. ∎

Example 4.6 *Let us continue Example 4.1. Knuth's bijection applied to the word $w(M)$, gives the pair of tableaux*

$$(T_1, T_2) = \left(\begin{array}{|c|c}\hline 3 & \\ \hline 2 & \\ \hline 1 & \multicolumn{1}{|c|}{3} \\ \hline \end{array}\ ,\ \begin{array}{|c|cc}\hline 2 & & \\ \hline 1 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \right)$$

of conjugated shapes $\lambda_1 = (1, 1, 2)$ and $\lambda_1\tilde{\ } = (1, 3)$. The shape $\overline{\lambda_1} = (2, 3)$, complementary to the shape λ_1 of the tableau T_1 within the square 3×3, provides then the shape of the tableau $\overline{T_2}$. Filling in its entries by taking (in the reverse order) the complements within $\{1, 2, 3\}$ of the columns of $T2$, we obtain

$$\overline{T_2} = \begin{array}{|c|c|c}\hline 2 & 2 & \\ \hline 1 & 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array}\ .$$

The pair $\left(T_1, \overline{T_2}\right)$ is then a pair of complementary Young tableaux within the square $N \times N$. To get an element of $\mathcal{T}_N$, it suffices now to associate with each entry i of T_1 (resp. of T_2) the letter δ_i (resp. χ_i) of the alphabet δ (resp. χ). The element of $\mathcal{T}_N$ associated in such a way with the initial matrix M of $\mathcal{M}_{N\times N}(\{0,1\})$, will be denoted by $\Phi(M)$ in the sequel.

Since the mapping $T_2 \to \overline{T_2}$ is one to one, it is now clear that we constructed in such a way a bijection Φ between $\mathcal{M}_N(\{0,1\})$ and $\mathcal{T}_N$. The problem is now to explore the properties of this bijection in order to be able to get some interesting enumerative consequences.

Example 4.7 *Let us finish the previous example 4.6 which was itself a continuation of Example 4.1. The element of $\mathcal{T}_3$ which is associated with the pair $(T_1, \overline{T_2})$ is given below:*

$$\Phi(M) \quad = \quad \begin{array}{|c|c|c|} \hline \delta_3 & \chi_2 & \chi_3 \\ \hline \delta_2 & \chi_2 & \chi_1 \\ \hline \delta_1 & \delta_3 & \chi_1 \\ \hline \end{array}$$

where M stands for the matrix introduced in Example 4.1.

4.3 Symmetry properties of our bijection

We will present here a very strong symmetry property of our bijection Φ. To this purpose, we give first another method for constructing it, presented below.

1. Construct again the two row array A_N which is the sequence of the N^2 pairs (i,j) of $\{1,\ldots,N\}\times\{1,\ldots,N\}$ taken in the lexicographic order, i.e.

$$A_N = \left(\begin{array}{cccccccccc} 1 & \ldots & 1 & 2 & \ldots & 2 & \ldots\ldots & N & \ldots & N \\ 1 & \ldots & N & 1 & \ldots & N & \ldots\ldots & 1 & \ldots & N \end{array} \right).$$

   Select then in A_N all the pairs corresponding to the 1's of M. We obtain then a first word $w_1(M)$ by reading the second component of the selected entries.

2. Construct then the two row array B_N which is equal to the sequence of the N^2 pairs (i,j) of $\{1,\ldots,N\}\times\{1,\ldots,N\}$ taken in the antilexicographic order (that is to say the lexicographic order with respect to the second entry), i.e.

$$B_N = \left(\begin{array}{cccccccccc} 1 & \ldots & N & 1 & \ldots & N & \ldots\ldots & 1 & \ldots & N \\ 1 & \ldots & 1 & 2 & \ldots & 2 & \ldots\ldots & N & \ldots & N \end{array} \right).$$

   Select in this array all pairs corresponding to the 0's of M. We obtain then a second word $w_2(M)$ by reading the first component of the selected entries.

One construct then two Young tableaux (T_1', T_2') by applying the column bumping process to the two previous words $w_1(M)$ and $w_2(M)$. It appears that these tableaux are exactly the two Young tableaux obtained by the bijection Φ, constructed in the previous subsection, applied to the matrix M.

Example 4.8 *This example continues again Example 4.6. Since the first step of the both ways to construct bijection Φ is the same, we will get here*

$$A_3 = \left(\begin{array}{ccccccccc} 1 & 1 & \boxed{1} & \boxed{2} & 2 & 2 & 3 & \boxed{3} & \boxed{3} \\ 1 & 2 & \boxed{3} & \boxed{1} & 2 & 3 & 1 & \boxed{2} & \boxed{3} \end{array} \right).$$

For the second array, we have in the same way:

$$B_3 = \left(\begin{array}{ccccccccc} \boxed{1} & 2 & \boxed{3} & \boxed{1} & \boxed{2} & 3 & 1 & \boxed{2} & 3 \\ \boxed{1} & 1 & \boxed{1} & \boxed{2} & \boxed{2} & 2 & 3 & \boxed{3} & 3 \end{array} \right) .$$

Hence we get

$$w_1(M) = (3,1,2,3) \ , \quad w_2(M) = (1,3,1,2,2) \ .$$

The column bumping process applied to $w_1(M)$ and $w_2(M)$ gives us then immediately the two following Young tableaux:

$$(T_1', T_2') = \left(\begin{array}{|c|c|} \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline \end{array} \ , \ \begin{array}{|c|c|c|} \hline 2 & 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 & 3 \\ \hline \end{array} \right) = \left(T_1, \overline{T_2} \right) \ .$$

We are now in position to state the following proposition which expresses the main symmetry property of our construction.

Proposition 4.9 *For every matrix M of $\mathcal{M}_{N\times N}(\{0,1\})$, one has :*

$$\Phi(M) = (T_1', T_2')$$

Proof – We will not give here the proof of this important result sicne it is rather technical. It is just worthwhile to note that our proof is based on the explicitation of the strong relations that exist between the Greene's invariants of the two words $w_1(M)$ and $w_2(M)$. ■

4.4 Some specializations of Barett's formula

As a consequence of the bijection, we can get some interesting combinatorial identities for several special cases of Barett's formula. It is for instance obvious to see that our bijection leads immediately to the identity

$$F(\chi,\delta) + F(\delta,\chi) = \sum_{k=0}^{N^2} \binom{N^2}{k} \delta^k \chi^{N^2-k}$$

in the situation where one substitutes in the symmetric function F all δ_i and χ_i by a single value respectively equal to δ and χ.

In a more interesting level, it is also possible to use our bijection in order to get an explicit combinatorial interpretation (which was still an open problem) of the coefficients of the polynomial of two variables resulting from the substitution of the k first variables δ_i and χ_i by single values and of all last $N-k$ variables δ_j and χ_j by 1. This interpretation is however rather long to explain: it will therefore be only presented in the final version of this paper.

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